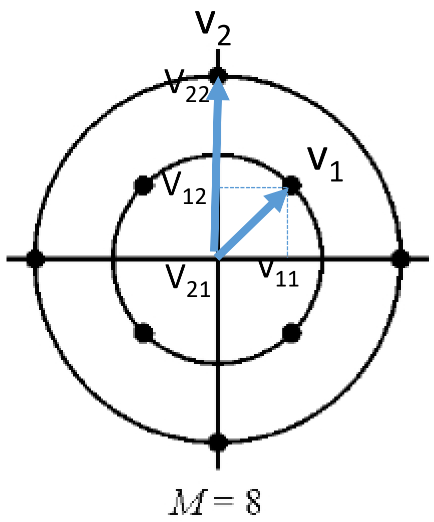
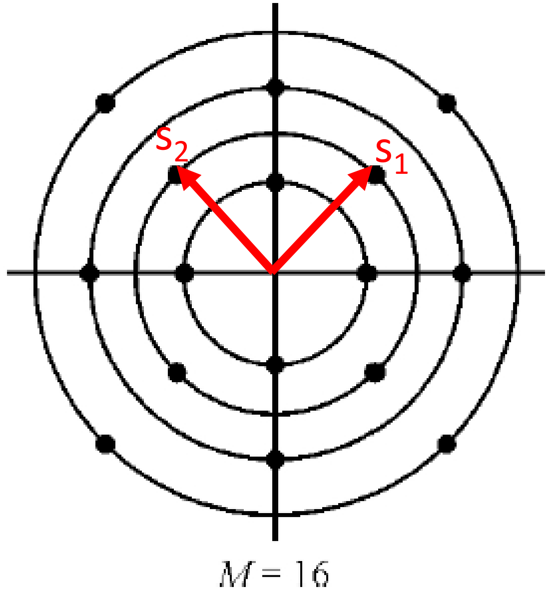
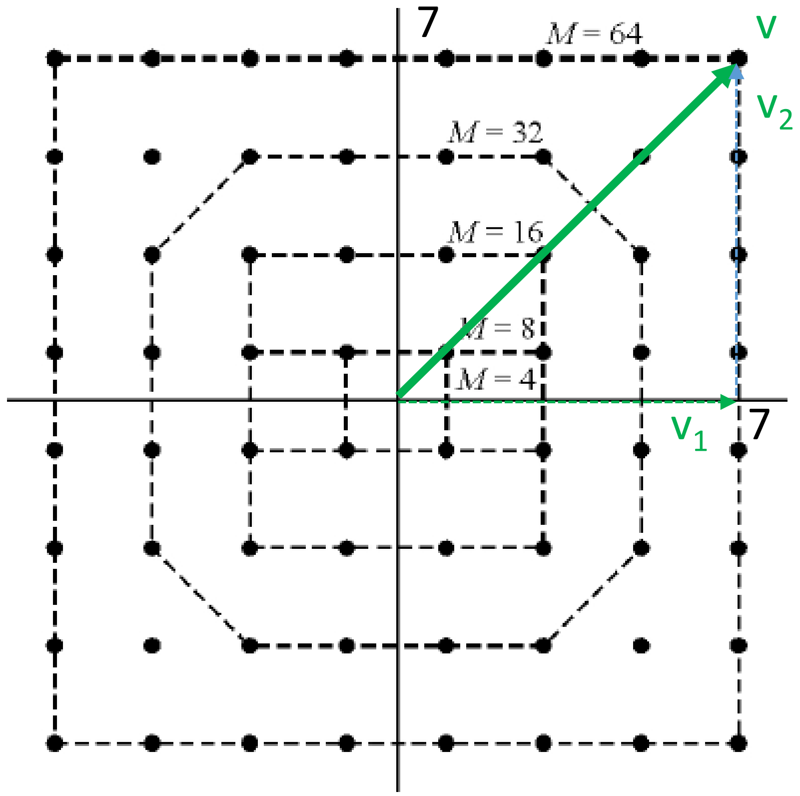
Quadrature Amplitude Modulation (QAM) (2)

➢ *Examples of constellation:*

We want to find:
Energy of the vector (symbol – signal)







Rectangular constellation

We want to find:
Correlation between vectors (symbols – signals)
Euclidean distance between vectors (symbols – signals)

4.4 SIGNAL SPACE REPRESENTATION

In this section, we demonstrate that signals have characteristics that are similar to vectors and develop a vector representation for signal waveforms

4.4.1 Vector Space Concepts

A vector v in n-dimensional space is characterized by its n components $[v_1, v_2, \ldots ,, v_n]$. It may also be represented as a linear combination of unit vectors or basis vectors e_i ,

$$v = \sum_{i=1}^{n} v_i e_i \tag{4.50}$$

The *inner product* of two n-dimensional vectors $v_1 = [v_{11}, v_{12}, \ldots, v_{1n}]$ and $v_2 = [v_{21}, v_{22}, \ldots ,v_{2n}]$ is

$$v_1 . v_2 = \sum_{i=1}^{n} v_{1i} v_{2i} \quad = v_{11}\ v_{21} + v_{12} v_{22} \text{ ; ak n=2} \tag{4.51}$$

Orthogonal vectors are if

$$v_1 . v_2 = 0 \tag{4.52}$$

The norm of a vector (simply its length) Kvadrat vzdialenosti vektora od pociatku = energia

$$\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2} \quad = \sqrt{v_1^2 + v_2^2} \quad ; ak\ n = 2; Pytagorova\ veta \tag{4.53}$$

A set of *m* vectors is said to be *orthonormal* if the vectors are orthogonal and each vector has a unit norm. A set of *m* vectors is said to be *linearly independent* if no one vector can be represented as a linear combination of the remaining vectors.

The norm square of the sum of two vectors may be expressed as

$$\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2 + 2 v_1 . v_2 \tag{4.54}$$

If v_1 and v_2 are ortogonal, then $\|v_1 + v_2\|^2 = \|v_1\|^2 + \|v_2\|^2$

4.4.3 *Orthogonal Expansions of Signals*

In this section, we develop a vector representation for signal waveforms, and, thus, we demonstrate an equivalence between a signal waveform and its vector representation.

Suppose that $s(t)$ is a deterministic real-valued signal with finite energy

$$E_S = \int_{-\infty}^{\infty} [s(t)]^2 dt \tag{4.65}$$

There exists a set of function $\{f_n(t), n = 1,2, \ldots, K\}$ that are orthonormal. We may approximate the signal $s(t)$ by a weighted linear combination of these functions, i.e.,

$$\hat{s}(t) = \sum_{k=1}^{K} s_k f_k(t) \tag{4.66}$$

where s_k are the coefficients in the approximation of $s(t)$.

The approximation error is

$$e(t) = s(t) - \hat{s}(t) \tag{4.67}$$

Let us select the coefficients $\{s_k\}$ so as to minimize the energy E_e of the approximation error.

$$E_e = \int_{-\infty}^{\infty} [s(t) - \hat{s}(t)]^2 dt = \int_{-\infty}^{\infty} [s(t) - \sum_{k=1}^{K} s_k f_k(t)]^2 dt \tag{4.68}$$

The optimum coefficients in the series expansion of $s(t)$ may be found by differentiating Equation 4.64 with respect to each of the coefficients $\{s_k\}$ and setting the first derivatives to zero.

Under the condition that $E_{min} = 0$ we may expres s(t) as

$$s(t) = \sum_{k=1}^{K} s_k f_k(t) \tag{4.69}$$

When every finite energy signal can be represented by a series expansion of the form (4.69) for which $E_{min} = 0$, the set of orthonormal functions $\{f_n(t)\}$ is said to be ***complete***.

Once we have constructed the set of orthonormal waveforms $\{f_n(t)\}$, we can express the M signals $\{s_n(t)\}$ as linear combinations of the $\{f_n(t)\}$.

Thus we may write

$$s_k(t) = \sum_{n=1}^{N} s_{kn} f_n(t) \tag{4.77}$$

and

$$E_k = \int_{-\infty}^{\infty} [s_k(t)]^2 dt = \sum_{n=1}^{N} s_{kn}^2 = \|s_k\|^2 \tag{4.78}$$

Based on the expression in Equation 4.77, each signal may be represented by the vector

$$s_k(t) = \{s_{k1} s_{k2} \dots s_{kN}\} \tag{4.79}$$

or, equivalently as a point in N-dimensional signal space with coordinates $\{s_{ki}, i = 1,2, \dots, N\}$.

The energy in the kth signal is simply the square of the length of the vector or, equivalently, the square of the Euclidean distance from the origin to the point in the N-dimensional space. Thus, any signal can be represented geometrically as a point in the signal space spanned by the orthonormal functions $\{f_n(t)\}$.

We have demonstrated that a set of M finite energy waveforms $\{s_n(t)\}$ can be represented by a weighted linear combination of orthonormal functions $\{f_n(t)\}$ of dimensionality $N \leq M$. The functions $\{f_n(t)\}$ are obtained by applying the Gram-Schmidt orthogonalization procedure on $\{s_n(t)\}$. It should be emphasized, however, that the functions $\{f_n(t)\}$ obtained from Gram-Schmidt procedure are not unique. If we alter the order in which the orthogonalization of the signals $\{s_n(t)\}$ is performed, the orthonormal waveforms will be different and the corresponding vector representation of the signals $\{s_n(t)\}$ will depend on the choice of the orthonormal functions $\{f_n(t)\}$. Nevertheless, the vectors $\{s_n\}$ will retain their geometrical configuration and their lengths will be invariant to the choice of orthonormal functions $\{f_n(t)\}$.

The orthogonal expansions described above were developed for real-valued signal waveforms.

Finally, let us consider the case in which the signal waveforms are band-pass and represented as

$$s_m(t) = Re[s_{lm}(t)e^{j2\pi f_c t}] \tag{4.80}$$

where $\{s_{lm}(t)\}$ denote the equivalent low-pass signals. Signal energy may be expressed either in terms of $s_m(t)$ or $s_{lm}(t)$, as

$$E_m = \int_{-\infty}^{\infty} s_m^2(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} |s_{lm}(t)|^2 dt \tag{4.81}$$

The similarity between any pair of signal waveforms, say $s_m(t)$ and $s_k(t)$ is measured by the ***normalized cross correlation***

$$\frac{1}{\sqrt{E_m E_k}}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt = Re\{\frac{1}{2\sqrt{E_m E_k}}\int_{-\infty}^{\infty} s_{lm}(t)s_{lk}^*(t)dt\} \tag{4.82}$$

We define the complex-valued cross-correlation coefficient ρ_{km} as

$$\rho_{km} = \frac{1}{2\sqrt{E_m E_k}}\int_{-\infty}^{\infty} s_{lm}^*(t)s_{lk}(t)dt \tag{4.83}$$

then

$$Re(\rho_{km}) = \frac{1}{\sqrt{E_m E_k}}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt \tag{4.84}$$

or, equivalently

$$Re(\rho_{km}) = \frac{s_m \cdot s_k}{\|s_m\|\|s_k\|} = \frac{s_m \cdot s_k}{\sqrt{E_m E_k}} \tag{4.85}$$

The cross-correlation coefficients between pairs of signal waveforms or signal vectors comprise one set of parameters that characterize the similarity of a set of signals. Another related parameter is the ***Euclidean distance between*** a pair of signals

$$d_{km}^{(e)} = \|s_m - s_k\| = \{\int_{-\infty}^{\infty}[s_m(t) - s_k(t)]^2 dt\}^{\frac{1}{2}}$$

$$= \{E_m + E_k - 2\sqrt{E_m E_k} Re(\rho_{km})\}^{\frac{1}{2}} \quad (4.86)$$

When $E_m = E_k = E$ for all m and k, this expression simplifies to

$$d_{km}^{(e)} = \{2E[1 - Re(\rho_{km})]\}^{\frac{1}{2}} \quad (4.87)$$

Thus, the Euclidean distance is an alternative measure of the similarity (or dissimilarity) of the set of signal waveforms or the corresponding signal vectors.

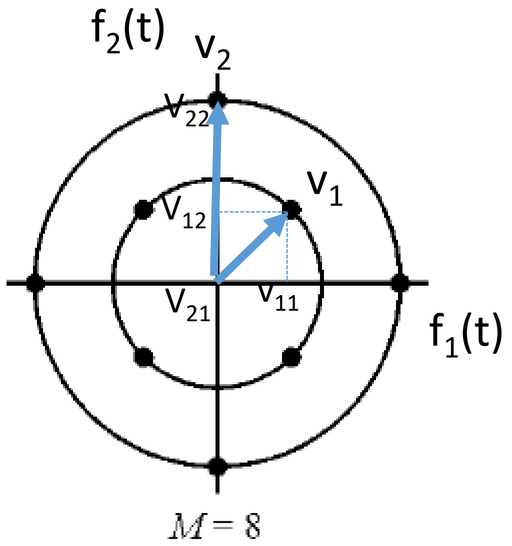
In the following section, we describe *digitally modulated signals* and make use of the signal space representation for such signals. We shall observe that digitally modulated signals, which are classified as *linear*, are conveniently expanded in terms of two orthonormal basis functions of the form

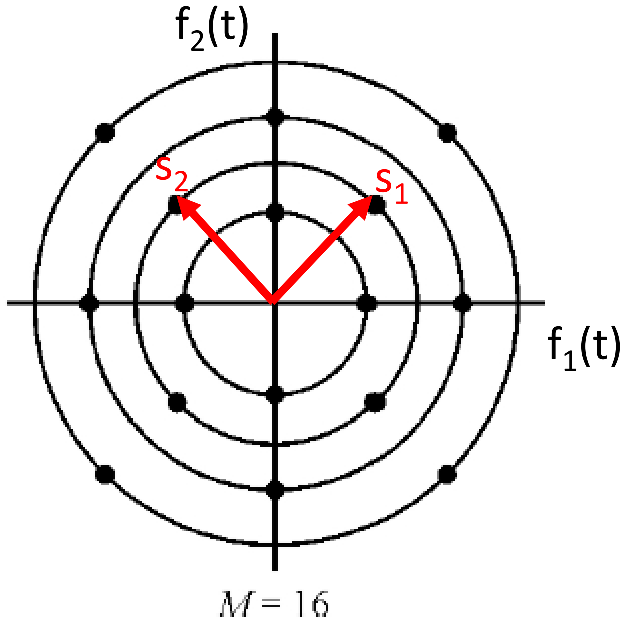
$$f_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t$$

$$f_2(t) = -\sqrt{\frac{2}{T}} \sin 2\pi f_c t \quad (4.88)$$

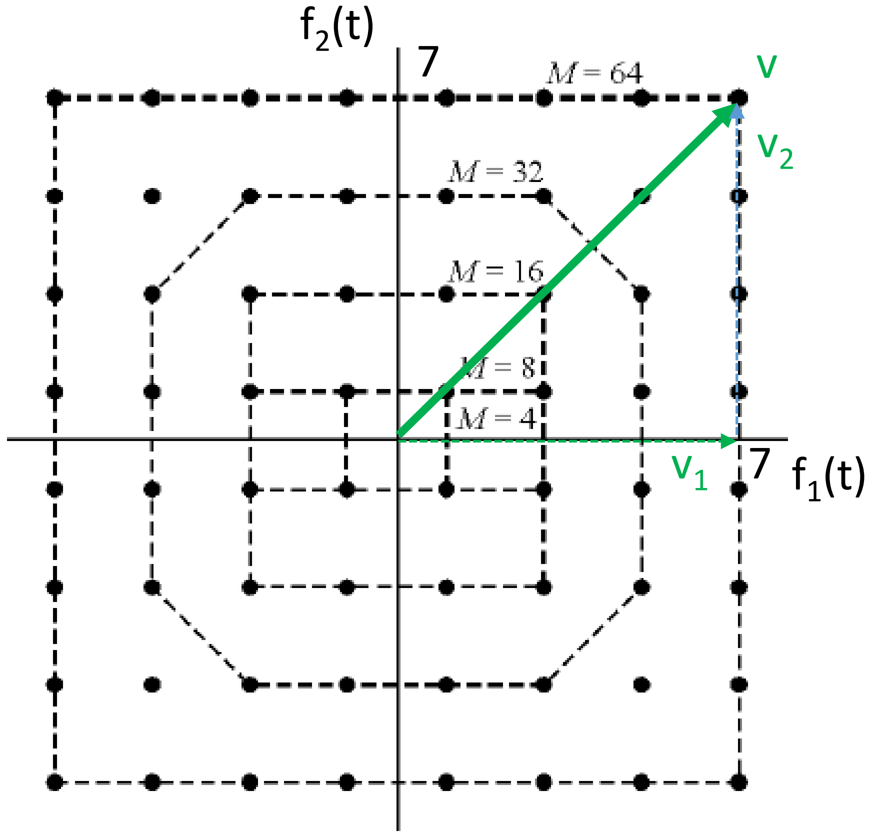
Quadrature Amplitude Modulation (QAM) (2)

➤ *Examples of constellation:*





We want to find:
Energy of the vector (symbol – signal)



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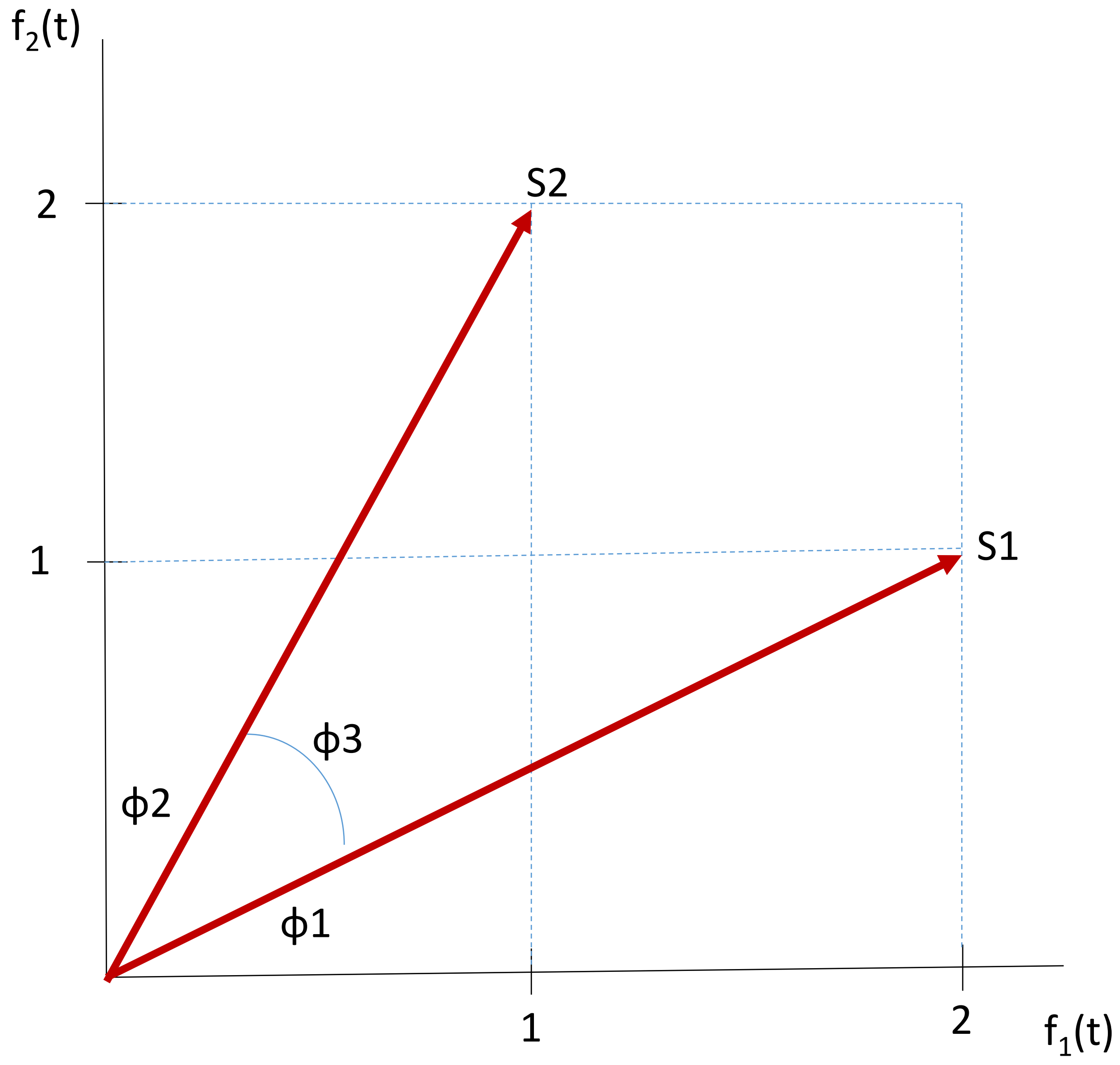
Euclidean distance between vectors (symbols – signals)

Rectangular constellation



Postup 1:

$\mathrm{Sin}\ \phi 1 = \frac{1}{\sqrt{5}} = 0{,}447213595499958$

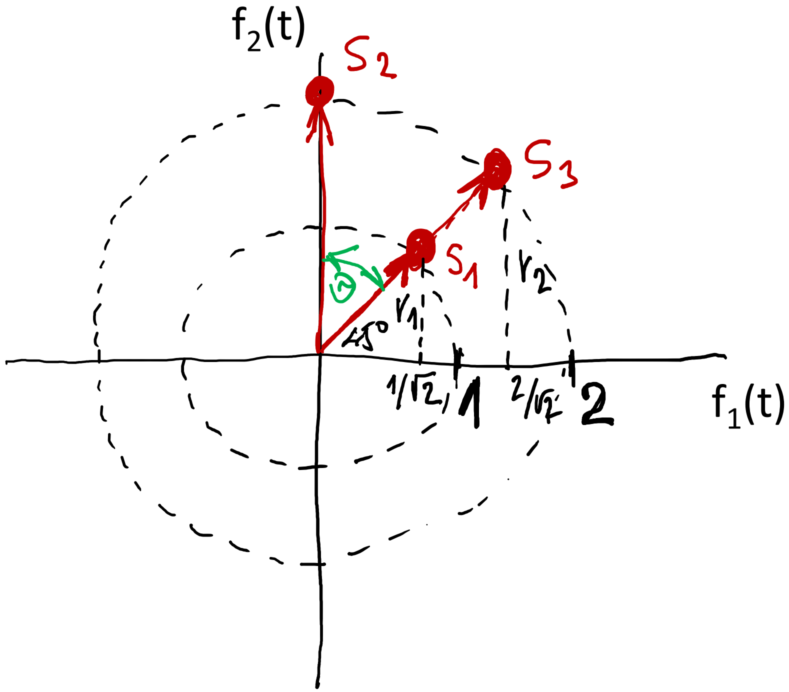
$\phi 1 = \phi 2 = 26{,}565051177^{o}$

$\phi 3 = 36{,}86989764584402^{o}$

$\rho_{S1S2} = \mathrm{Cos}\ \phi 3 = 0.8$

Postup 2:

$$\rho_{S1S2} = \frac{s_1 . s_2}{\sqrt{E_{s1} E_{s2}}} = \frac{2.1+1.2}{\sqrt{5.5}} = \frac{4}{5} = 0.8$$



a) $\rho_{s_1 s_2} = ?$

$\sin 45^\circ = \frac{r_1}{1} \Rightarrow r_1 = \sin 45^\circ = \frac{1}{\sqrt{2}} = 0{,}707$

$$\rho_{s_1 s_2} = \frac{s_1 \cdot s_2}{\sqrt{E_{s_1} E_{s_2}}} = \frac{s_{11} \cdot s_{21} + s_{21} s_{22}}{\sqrt{E_{s_1} E_{s_2}}} = \frac{\frac{1}{\sqrt{2}} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 2}{\sqrt{1 \cdot 4}} = \frac{1}{\sqrt{2}}$$

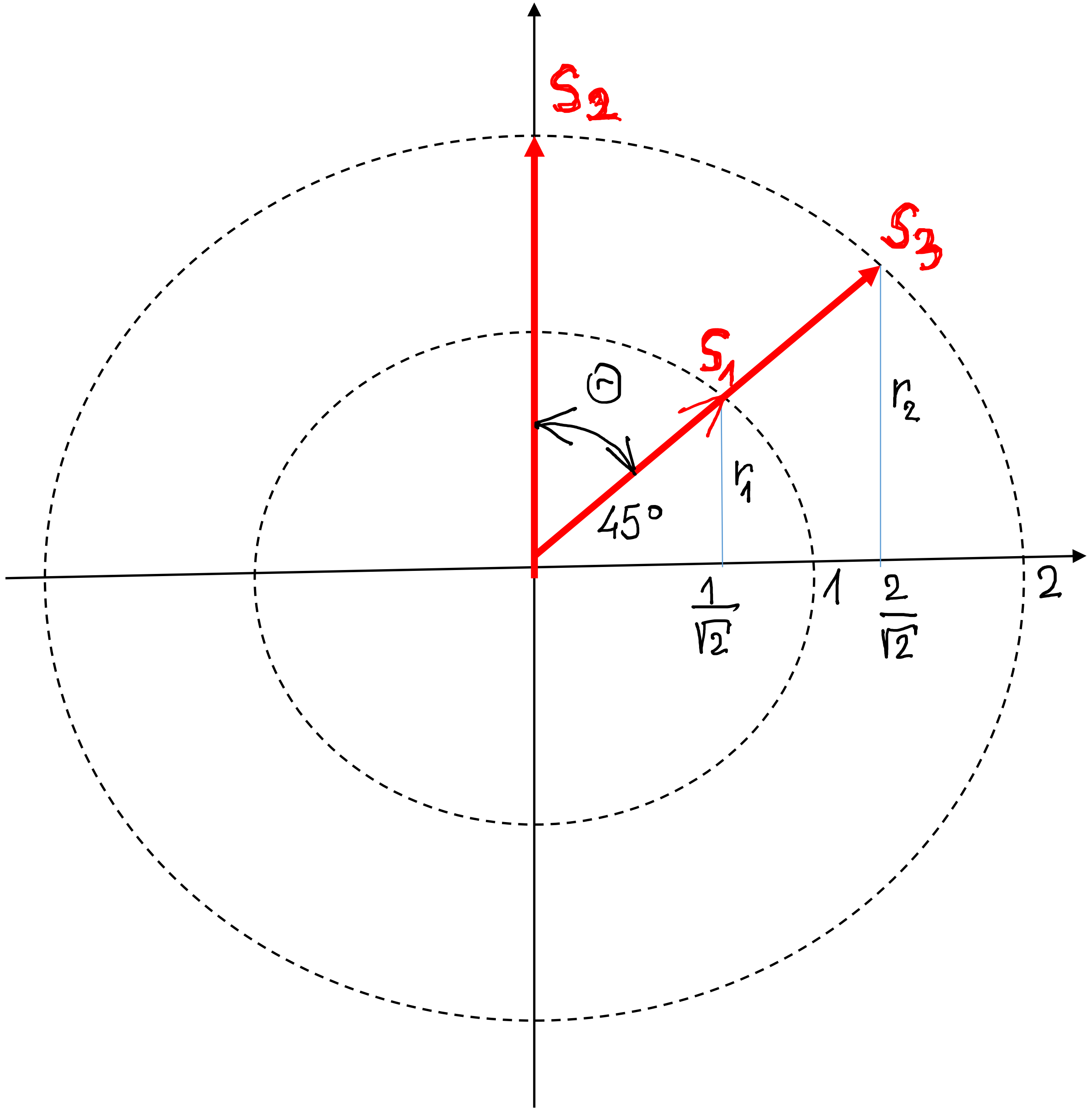
resp: $\rho_{s_1 s_2} = \cos \theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$

b) $\rho_{s_3 s_2} = ?$

$\sin 45^\circ = \frac{r_2}{2} \Rightarrow r_2 = 2 \cdot \sin 45^\circ = \frac{2}{\sqrt{2}}$

$$\rho_{s_3 s_2} = \frac{s_3 \cdot s_2}{\sqrt{E_3 E_2}} = \frac{\frac{2}{\sqrt{2}} \cdot 0 + \frac{2}{\sqrt{2}} \cdot 2}{\sqrt{2^2 \cdot 2^2}} = \frac{4}{\sqrt{2 \cdot 16}} = \frac{1}{\sqrt{2}}$$

resp: $\rho_{s_3 s_2} = \cos \theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$



a) $\rho_{s_1 s_2} = ?$

$$\sin 45^\circ = \frac{r_1}{1} \Rightarrow r_1 = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

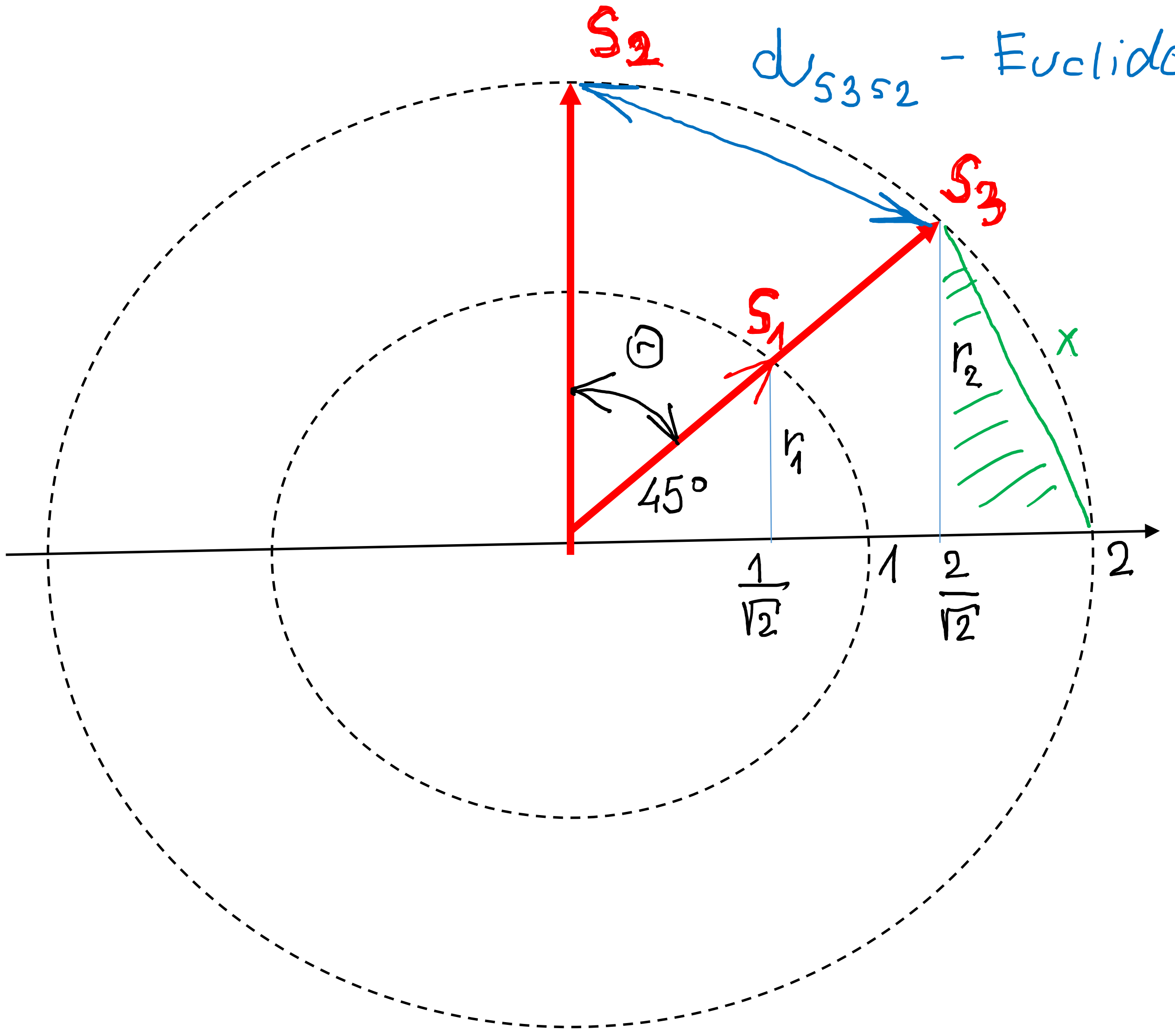
$$\rho_{s_1 s_2} = \frac{s_1 \cdot s_2}{\sqrt{E_{s_1} E_{s_2}}} = \frac{1/\sqrt{2} \cdot 0 + \frac{1}{\sqrt{2}} \cdot 2}{\sqrt{1 \cdot 4}} = \frac{1}{\sqrt{2}}$$

or $\rho_{s_1 s_2} = \cos \theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$

b) $\rho_{s_3 s_2} = ?$

$$\rho_{s_3 s_2} = \frac{1}{\sqrt{2}}$$

$d_{s_3 s_2}$ - Euclidean distance



a) $d_{s_3 s_2} = \sqrt{\left(E_{s_3} + E_{s_2} - 2\sqrt{E_{s_3} E_{s_2}}\, \rho_{s_3 s_2}\right.}$

$$d_{s_3 s_2} = \sqrt{4 + 4 - 2\sqrt{4 \cdot 4} \cdot \frac{1}{\sqrt{2}}}$$

$$d_{s_3 s_2} = \sqrt{8 - \frac{8}{\sqrt{2}}}$$

b)

$$x = \sqrt{\left(2 - \frac{2}{\sqrt{2}}\right)^2 + \left(\frac{2}{\sqrt{2}}\right)^2}$$

$$x = \sqrt{4 - 2\left(2 \cdot \frac{2}{\sqrt{2}}\right) + 2 + 2}$$

$$x = \sqrt{8 - \frac{8}{\sqrt{2}}}$$

Problem 4

Determine the correlation coefficient ρ_{km} among the four signal waveforms $\{s_i(t)\}$ shown in Figure in Problem 2, and the corresponding Euclidean distances.

Solution

For real-valued signals the correlation coefficients are given by:

$$\rho_{km} = \frac{1}{\sqrt{E_m E_k}}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt$$

And the Euclidean distances by:

$$d_{km}^{(e)} = \{E_m + E_k - 2\sqrt{E_m E_k}Re(\rho_{km})\}^{\frac{1}{2}}$$

For the signals in problem:

$$E_1 = 2, E_2 = 2, E_3 = 3, E_4 = 3$$

$$\rho_{12} = 0, \rho_{13} = \frac{2}{\sqrt{6}}, \rho_{12} = -\frac{2}{\sqrt{6}},$$

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$$\rho_{23} = 0, \rho_{24} = 0,$$

$$\rho_{34} = -\frac{1}{3},$$

and

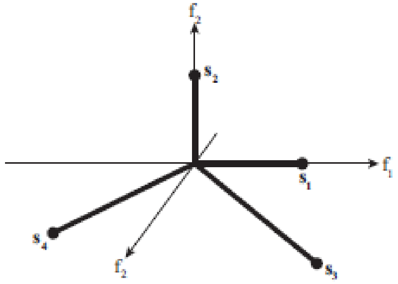
$$d_{12}^{(e)} = 2, d_{12}^{(e)} = \sqrt{2+3-2\sqrt{6}\frac{2}{\sqrt{6}}} = 1, d_{12}^{(e)} = \sqrt{2+3+2\sqrt{6}\frac{2}{\sqrt{6}}} = 3$$

13 14

$$d_{23}^{(e)} = \sqrt{2+3} = \sqrt{5}, d_{24}^{(e)} = \sqrt{5}$$

$$d_{34}^{(e)} = \sqrt{3+3+2\frac{1}{3}} = 2\sqrt{2}$$

Since the dimensionality of the signal space is N=3, each signal is described by three components. The signal is characterized by the vector $s_1(t)=(\sqrt{2},0,0)$. Similarly, the signals $s_2(t)$, $s_3(t)$and $s_4(t)$ are characterized by the vectors $s_2(t)=(0,\sqrt{2},0), s_3(t)=(\sqrt{2},0,1)$ and $s_4(t)=(-\sqrt{2},0,1)$ respectively. These vectors are shown in Figure:



Their lengths are $\|s_1\| = \sqrt{2}$, $\|s_2\| = \sqrt{2}$, 3 $\|s_3\| = \sqrt{3}$, and $\|s_4\| = \sqrt{3}$ and the corresponding signal energies $E_k = \|s_1\|^2$, $k = 1,2,3,4$.

$$Re(\rho_{km}) = \frac{s_m.s_k}{\|s_m\|\|s_k\|} = \frac{s_m.s_k}{\sqrt{E_m E_k}}$$

Example:

$$s_1 . s_3 = (\sqrt{2},0,0).(\sqrt{2},0,1) = \sqrt{2}.\sqrt{2} + 0.0 + 0.1 = 2$$

$$E_1 = ||s_1||^2 = 2 \qquad E_3 = ||s_3||^2 = 3$$

$$\rho_{13} = \frac{s_1 . s_3}{\sqrt{E_1 . E_3}} = \frac{2}{\sqrt{6}}$$